MATH 3A WEEK III
VECTOR SPACES

PAUL L. BAILEY

1. Introduction to Abstraction

Mathematics may be described as the study of *patterns*; not only patterns that arise is the real universe, but of all *possible* patterns that may arise in *any* universe.

Abstraction is the process of identifying the key attributes in a system and extracting them; these key attributes may be called *defining properties*. Then one investigates the implications of these attributes, disembodied from the system which originally motivated their investigation. The investigation is rigorous, and all conclusions drawn are supported purely by logic, not by the physical world. This support is called *proof*. Together, abstraction and proof are the *mathematical method*; they allow us to isolate and study patterns.

This method has three main benefits.

(1) The proof aspect ensures that all claims made are true (given the assumptions); additionally, it is often the proof itself that illuminates the situation so that we understand it more fully.
(2) The abstraction allows one to see exactly how the identified key attributes lead to the behavior being displayed in the motivating system.
(3) All results apply to *any* system which obeys the defining properties.

For example, the set $\mathbb{R}$ of real numbers has many attributes which effect how we think of them, for example:

- Order
- Distance
- Algebra

Although these attributes are interelated in the case of the real numbers, by looking at systems which *a priori* have only order, distance, or algebra, we obtain a better understanding of how these attributes effect our understanding of the real numbers.

In particular, we will see that in the case of vector spaces, the abstracted properties are so universal that they appear repeatedly in various forms, and we benefit from the abstraction we now make.

We use standard English and assume the laws of logic as determined by truth tables, a preliminary knowledge of set theory, and the basic algebraic properties of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We make no other assumptions, and proceed to develop a complete theory from this starting point. In our examples and applications, however, we may assume previous knowledge, such as calculus.

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2. Vector Spaces

Definition 1. A *vector space* over $\mathbb{R}$ is a set V together with a pair of operations

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{R} \times V \to V,$$

called *vector addition* and *scalar multiplication* respectively, satisfying

(V1) $v + w = w + v$ for all $v, w \in V$;
(V2) $v + (w + x) = (v + w) + x$ for all $v, w, x \in V$;
(V3) there exists $0_V \in V$ such that $v + 0_V = v$ for all $v \in V$;
(V4) for every $v \in V$ there exists $w \in V$ such that $v + w = 0_V$;
(V5) $1 \cdot v = v$ for every $v \in V$;
(V6) $(ab)v = a(bv)$ for every $v \in V$ and $a, b \in \mathbb{R}$;
(V7) $a(v + w) = av + aw$ for every $v, w \in V$ and $a \in \mathbb{R}$;
(V8) $(a + b)v = av + bv$ for every $v \in V$ and $a \in \mathbb{R}$.

The elements of V are called *vectors*.

Remark 1. The $\cdot$ is usually suppressed in this notation (as in **(V6)**, **(V7)**, and **(V8)**); scalar multiplication is instead denoted by juxtiposition.

Remark 2. Elements are added two at a time. However, because of property **(V2)**, parentheses are useless to distinguish the order of addition. That is,

$$v_1 + \cdots + v_n$$

makes sense without inserting parentheses to denote the order in which the elements are added, since any order gives the same result.

Remark 3. In the absense of parentheses, the operations of vector addition and scalar multiplication are written with $\cdot$ having higher precedence over $+$. For example, if $a, b \in \mathbb{R}$ and $v, w \in V$, $ax + by$ means $(ax) + (by)$.

3. Examples of Vector Spaces

Example 1. Let $V = \{0\}$. Then V is called the *trivial* vector space.

Example 2. Let $\mathbb{R}$ be the set of real numbers, together with their standard addition and multiplication.
Then $\mathbb{R}$ is a vector space.

Example 3. Let $\mathbb{R}^n$ be the set of ordered n-tuples of real numbers, together with vector addition and scalar multiplication as defined previously.
Then $\mathbb{R}^n$ is a vector space.

Example 4. Let $V \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ under our previous interpretation, together with vector addition and scalar multiplication from $\mathbb{R}^n$.
Then V is a vector space.

Example 5. Let $\mathcal{M}_{m\times n}$ be the set of $m \times n$ matrices with real entries, together with matrix addition and scalar multiplication as defined previously.
Then $\mathcal{M}_{m\times n}$ is a vector space.

Example 6. Let $I \subset \mathbb{R}$ be an open interval. Let $\mathcal{F}(I) = \{f : I \to \mathbb{R}\}$ be the set of all functions from I into $\mathbb{R}$. Note that we have specified such a function if we specify its value at every point in I. Define addition and scalar multiplication by

$$(f + g)(t) = f(t) + g(t) \quad \text{where } f, g \in \mathcal{F}(I), t \in I;$$
$$(af)(t) = a(f(t)) \quad \text{where } f \in \mathcal{F}(I), a \in \mathbb{R}, t \in I.$$

Then $\mathcal{F}(I)$, together with these operations, is a vector space.

Example 7. Let $\mathcal{P}$ denote the set of all polynomial functions with real coefficients, and for each $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the set of all polynomial functions of degree less than or equal to n with real coefficients:

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1x + \cdots + a_nx^n \text{ where } a_i \in \mathbb{R}; n \in \mathbb{N}\};$$
$$\mathcal{P}_n = \{f \in \mathcal{P} \mid \deg(f) \leq n\}.$$

Define addition and scalar multiplication on these sets as in the case of $\mathcal{F}(I)$.
Then $\mathcal{P}$ and $\mathcal{P}_n$, together with these operations, are vector spaces.

Example 8. Let V and W be vector spaces. Define vector addition and scalar multiplication on the cartesian product $V \times W$ by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2);$$
$$a(v, w) = (av, aw).$$

Then $V \times W$, together with these operations, is a vector space. Indeed, this is exactly how $\mathbb{R}^n$ is constructed from $\mathbb{R}$.

4. Properties of Vector Addition and Scalar Multiplication

Proposition 1. *Let V be a vector space. Suppose that there exists $0_1, 0_2 \in V$ such that $v + 0_1 = v$ and $v + 0_2 = v$ for every $v \in V$. Then $0_1 = 0_2$.*

Proof. We have $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$. □

Remark 4. This proposition says the *the additive identity is unique.*

Proposition 2. *Let V be a vector space and let $v \in V$. Suppose that there exists w_1 and w_2 such that $v + w_1 = 0_V$ and $v + w_2 = 0_V$. Then $w_1 = w_2$.*

Proof. Since $v + w_1 = 0_V$, we have $w_2 + (v + w_1) = w_2 + 0_V$. By **(V2)** on the left and **(V3)** on the right, we have, $(w_2 + v) + w_1 = w_2$. By property **(V1)**, $(v + w_2) + w_1 = w_2$, and by assumption on w_2, this gives $0_V + w_1 = w_2$. By property **(V2)**, $w_1 + 0_V = w_2$, and finally by property **(V3)** we obtain $w_1 = w_2$. □

Remark 5. This proposition says that *additive inverses are unique.* We denote the unique additive inverse of v by $-v$. We shorten $w + (-v)$ to $w - v$.

Proposition 3. (Cancellation Law)
Let V be a vector space and let $v, w, x \in V$. Then $v + x = w + x \Rightarrow v = w$.

Proof. Add $-x$ to both sides. □

Proposition 4. *Let V be a vector space. Let $v \in V$ and $a \in \mathbb{R}$. Then*

- **(a)** $0v = 0_V$;
- **(b)** $a0_V = 0_V$;
- **(c)** $av = 0_V \Rightarrow a = 0$ *or* $v = 0_V$;
- **(d)** $(-1)v = -v$;
- **(e)** $(-a)v = -(av)$.

Proof.

(a) We have $v + (0v) = (1v) + (0v) = (1 + 0)v = 1v = v$; thus $0v$ acts like the additive identity, so it must be the additive identity by uniqueness.

(b) If $a = 0$, the result follows from **(a)**, so assume $a \neq 0$. We have $v+(a0_V) = (aa^{-1})v + a0_V = a(a^{-1}v) + a0_V = a(a^{-1}v + 0_V) = a(a^{-1}v) = v$; thus $v + (a0_V)$ acts like the additive identity, so it must be the additive identity.

(c) Exercise.

(d) Since $1v = v$, this is a special case of **(d)**.

(e) We have $av + (-a)v = (a + (-a))v = 0v = 0_V$; thus $(-a)v$ acts like the additive inverse of a, so it must be the additive inverse of v by uniqueness. □

Remark 6. We now drop the subscript from 0_V and just write 0. We distinguish this from $0 \in \mathbb{R}$ by context.

5. Subspaces

Definition 2. Let V be a vector space.

A *subspace* of V is a subset $W \subset V$ which satisfies:

(S0) $0 \in W$;
(S1) $x, y \in W \Rightarrow x + y \in W$;
(S2) $a \in \mathbb{R}, x \in W \Rightarrow ax \in W$.

If W is a subspace of V, this is denoted by $W \leq V$.

Remark 7. If $W \leq V$, then W is a vector space under the same operations of addition and scalar multiplication. If is clear that if $U \leq W$ and $W \leq V$, then $U \leq W$.

Remark 8. Consider the condition

(SE) W is nonempty.

In the presence of **(S1)** and **(S2)**, we see that **(SE)** is equivalent to **(S0)**. Clearly, if $0 \in W$, then W is nonempty. On the other hand, suppose that W is nonempty. Then W contains a vector, say $w \in W$. By **(S2)**, we see that $0w = 0 \in W$.

Example 9. Let V be a vector space; then there exists a zero element $0 \in V$, and $\{0\} \leq V$; this is called the *trivial* subspace.

Example 10. Let $V = \mathbb{R}^3$ and let $W = \{(x_1, x_2, x_3) \in V \mid x_1 + x_2 + x_3 = 0\}$. Show that $W \leq V$.

Solution. To show that W is a subspace of V, we verify the three properties of being a subspace.

(S0) We wish to show that $0_V \in W$. Since $0 + 0 + 0 = 0$, we see that $(0, 0, 0) = 0_V \in W$.

(S1) We wish to show that the sum of two elements in W is also an element in W. Let $x, y \in W$. Then $x, y \in \mathbb{R}^3$, so $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_2)$ for some real numbers $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. By definition of W, $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$. Adding these equations and rearranging via properties **(V1)** and **(V2)** of V, we see that $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 0$. Thus $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ satisfies the defining condition of W, and must be an element of W.

(S2) We wish to show that any scalar multiple of an element in W is also an element in W. Let $x = (x_1, x_2, x_3) \in W$ and let $a \in \mathbb{R}$. Then $x_1 + x_2 + x_3 = 0$, so $a(x_1 + x_2 + x_3) = a0 = 0$; by distributing, we get $ax_1 + ax_2 + ax_3 = 0$. Thus $ax = (ax_1, ax_2, ax_3) \in W$. $\square$

Example 11. Fix $n \in \mathbb{N}$. Then $\mathcal{P}_n \leq \mathcal{P}$.

Example 12. Fix $m, n \in \mathbb{N}$ with $m \leq n$. Then $\mathcal{P}_m \leq \mathcal{P}_n$.

Example 13. A function $f : I \to \mathbb{R}$ is called *smooth* if it is infinitely differentiable on the interval I; that is, if derivatives of all orders exist and are continuous. The following are subspaces of $\mathcal{F}(I)$:

- $\mathcal{C}(I) = \{f \in \mathcal{F}(I) \mid f \text{ is continuous}\}$;
- $\mathcal{D}(I) = \{f \in \mathcal{F}(I) \mid f \text{ is smooth}\}$.

Note that $\mathcal{D}(I) \leq \mathcal{C}(I) \leq \mathcal{F}(I)$.

Proposition 5. *Let V be a vector space and let $W_1, W_2 \leq V$. Then $W_1 \cap W_2 \leq V$.*

Proof. We verify the three properties of a subspace.

(S0) Since $0 \in W_1$ and $0 \in W_2$, we have $0 \in W_1 \cap W_2$.

(S1) Let $x, y \in W_1 \cap W_2$. Then $x, y \in W_1$ and $x, y \in W_2$, so $x + y \in W_1$ and $x + y \in W_2$, because both of these sets are subspaces. Thus $x + y \in W_1 \cap W_2$.

(S2) Let $x \in W_1 \cap W_2$ and let $a \in \mathbb{R}$. Then $x \in W_1$ and $x \in W_2$, and since these are subspaces, we see that $ax \in W_1$ and $ax \in W_2$. Thus $ax \in W_1 \cap W_2$.

Therefore $W_1 \cap W_2 \leq V$. □

Remark 9. This argument generalizes so that the intersection of *any number* (even infinitely many) of subspaces is a subspace.

Definition 3. Let V be a vector space and let $X, Y \subset V$. Define the sum of these sets to be the subset of V given by

$$X + Y = \{x + y \mid x \in X, y \in Y\}.$$

Proposition 6. *Let V be a vector space and let $W_1, W_2 \leq V$. Then $W_1 + W_2 \leq V$.*

Proof. We verify the three properties of a subspace.

(S0) Since $0 \in W_1$ and $0 \in W_2$, we see that $0 = 0 + 0 \in W_1 + W_2$.

(S1) Let $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ so that $w_1 + w_2$ and $w_1' + w_2'$ are arbitrary members of $W_1 + W_2$. Then $(w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2') \in W_1 + W_2$, by properties **(V1)** and **(V2)** of V and by property **(S1)** of W_1 and W_2.

(S2) Let $w_1 \in W_1$ and $w_2 \in W_2$ so that $w_1 + w_2$ is an arbitrary member of $W_1 + W_2$ Let $a \in \mathbb{R}$. Then $a(w_1 + w_2) = aw_1 + aw_2 \in W_1 + W_2$, by property **(V7)** of V and property **(S2)** of W_1 and W_2. □

Remark 10. It follows that any finite sum of subspaces is a subspace.

6. Spans

Definition 4. Let V be a vector space and let $X \subset V$.

A *linear combination* of elements from X is an element of V of the form

$$a_1 v_1 + \cdots + a_n v_n \quad \text{where } a_i \in \mathbb{R} \text{ and } v_i \in X.$$

The *span* of X is denoted by $\mathrm{span}(X)$ and is defined by

$$\mathrm{span}(X) = \{v \in V \mid v \text{ is a linear combination from } X\}.$$

The span of the empty set is defined to be $\{0\}$.

Proposition 7. *Let V be a vector space and let $X, Y \subset V$. Then*

(a) $0 \in \mathrm{span}(X)$;
(b) $X \subset \mathrm{span}(X) \subset V$;
(c) $\mathrm{span}(X) = \mathrm{span}(\mathrm{span}(X))$;
(d) $X \subset Y \Rightarrow \mathrm{span}(X) \subset \mathrm{span}(Y)$;
(e) $X \subset \mathrm{span}(Y) \Rightarrow \mathrm{span}(X) \subset \mathrm{span}(Y)$;
(f) $X \leq V \Leftrightarrow X = \mathrm{span}(X)$.

Proof.

(a) If $x \in X$, then $0_V = 0x$ is a linear combination from X.

(b) To show that one set is a subset of another, it suffices to select an arbitrary element of the one set and show that it is in the other.

Let $x \in X$; then $x = 1x$, so x is a linear combination from X. Thus $x \in \mathrm{span}(X)$. Since x was arbitrary, we see that $X \subset \mathrm{span}(X)$.

Since V is closed under addition and scalar multiplication, every linear combination of vectors from V is also in V. Thus the span of any subset of V is contained in V, i.e., $\mathrm{span}(X) \subset V$.

(c) To show that two sets are equal, we show that each is contained in the other. By **(b)**, we know that $\mathrm{span}(X) \subset \mathrm{span}(\mathrm{span}(X))$, so we only need to show that $\mathrm{span}(\mathrm{span}(X)) \subset \mathrm{span}(X)$.

Let $y \in \mathrm{span}(\mathrm{span}(X))$. Then there exist vectors $y_1, \dots, y_n \in \mathrm{span}(X)$ and real number $a_1, \dots, a_n \in \mathbb{R}$ such that $y = \sum_{i=1}^{n} a_i y_i$. Each y_i is in $\mathrm{span}(X)$, so there exist a finite number (say m_i) of vectors $x_{i1}, \dots, x_{im_i} \in X$ and real numbers $b_{i1}, \dots, b_{im_i}$ such that $y_i = \sum_{j=1}^{m_i} b_{ij} x_{ij}$. Then $y = \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_i b_{ij} x_{ij}$ is a linear combination from X, so $y \in \mathrm{span}(X)$.

(d) Suppose that $X \subset Y$. Then every linear combination from X is a linear combination from Y. Thus $\mathrm{span}(X) \subset \mathrm{span}(Y)$.

(e) Suppose that $X \subset \mathrm{span}(Y)$. Then by **(d)**, $\mathrm{span}(X) \subset \mathrm{span}(\mathrm{span}(Y))$. But by **(c)**, $\mathrm{span}(\mathrm{span}(Y)) = \mathrm{span}(Y)$, so $\mathrm{span}(X) \subset \mathrm{span}(Y)$.

(f) To show an if and only if statement, we show implication in both directions.

($\Rightarrow$) Suppose that $X \leq V$. We know that $X \subset \mathrm{span}(X)$; we must show that $\mathrm{span}(X) \subset X$.

Let $x \in \mathrm{span}(X)$. Then x is a linear combination from X. This means that x is a finite sum of scalar multiples of elements of X. Since X is a subspace, it is closed under vector addition and scalar multiplication. Thus each scalar multiple is in X, and the sum of these elements of X is also in X. Thus $x \in X$.

($\Leftarrow$) Suppose that $X = \mathrm{span}(X)$. We have already noted that $0 \in X$. Let $x_1, x_2 \in X$ and let $a \in \mathbb{R}$. Then $x_1 + x_2$ and ax_1 are linear combinations from X, so they are in X since $X = \mathrm{span}(X)$. Since X satisfies **(S0)**, **(S1)**, and **(S2)**, we see that $X \leq V$. $\square$

Definition 5. Let V be a vector space and let $X \subset V$.
We say that X *spans* V if $V = \text{span}(X)$.
We say that V is *finitely generated* if it is spanned by a finite set of vectors.

Proposition 8. *Let V be a vector space and let $X, Y \subset V$.
If X spans V and $X \subset Y$, then Y spans V.*

Proof. Suppose that X spans V. Then every element of V is a linear combination of elements from X. But since $X \subset Y$, all such linear combinations are also linear combinations from Y. Thus Y spans V. □

Remark 11. Let V be a finitely generated vector space; this means that there exists a finite set of vectors, say X, such that X spans V. Suppose that Y is an infinite subset of V which spans V. The sets X and Y may not have any elements in common. The next proposition tells us than in spite of this, some finite subset of subset of Y spans V.

Proposition 9. *Let V be a finitely generated vector space and let $Y \subset V$ such that Y spans V. Then there exists a finite subset Z of Y such that Z spans V.*

Proof. Since V is finitely generated, there exists some finite set

$$X = \{x_1, \dots, x_n\} \subset V$$

such that $\text{span}(X) = V$. But $X \subset \text{span}(Y)$, so each of the vector $x_i \in X$ may be written as a linear combination of a finite number (say m_i) of vectors $z_{i1}, \dots, z_{im_j} \in Y$. Let $Z = \{z_{ij} \mid i = 1, \dots, n;\ j = 1, \dots, m_i\}$. Then Z is a finite set, and $X \subset \text{span}(Z)$, so $V = \text{span}(X) \subset \text{span}(Z)$. Since $\text{span}(Z) \subset V$, we have $\text{span}(Z) = V$. □

Example 14. Let $V = \mathbb{R}^3$ and let X be any set of vectors with the property that not all of them lie on the same plane. Then X spans V. Moreover, one can pick out a finite subset of X which spans V; soon we will see that one can choose this set with exactly three elements.

Example 15. Let $V = \mathbb{R}^n$. Let e_i denote the vector whose i^{th} entry is equal to one and whose other entries are equal to zero.
Let $X = \{e_1, \dots, e_n\}$. Then X spans V.

Example 16. Let $V = \mathcal{M}_{m\times n}$. Let M_{ij} denote the $m \times n$ matrix whose ij^{th} entry is equal to one and whose other entries are equal to zero.
Let $X = \{M_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n\}$. Then X spans V.

Example 17. Let $V = \mathcal{P}_n$ and let $X = \{1, x, x^2, \dots, x^n\}$. Then X spans $\mathcal{P}_n$.

Remark 12. One can show that $\text{span}(X)$ is *exactly* the intersection of all subspaces of V which contain X.

7. Linear Independence

Definition 6. Let V be a vector space and let $X \subset V$.

We say that X is *linearly independent*, or simply *independent*, if whenever $v_1, \dots, v_n \in V$ are distinct elements of X and $a_1, \dots, a_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} a_i v_i = 0 \Rightarrow a_i = 0 \text{ for } i = 1, \dots, n.$$

In this case we may also say that the vectors in X are linearly independent.

We say that X is *linearly dependent*, or simply *dependent*, if it is not independent. In this case we may also say that the vectors in X are linearly dependent.

Remark 13. If $X \subset V$ is dependent, there exists a *nontrivial dependence relation*; by this we mean that there exist distinct elements $x_1, \dots, x_n \in X$ and real numbers $a_1, \dots, a_n$, at least one of which is nonzero, such that

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Clearly, any set containing 0 is dependent.

Proposition 10. *Let V be a vector space and let $X, Y \subset V$.*
If Y is independent and $X \subset Y$, then X is independent.

Proof. Any nontrivial dependence relation among the elements of X would be a nontrivial dependence relation among the elements of Y. □

Proposition 11. *Let V be a vector space and let $X \subset V$. Then X is independent if and only if for every finite subset $B = \{x_1, \dots, x_n\} \subset X$ and every $x \in \operatorname{span}(B)$ there exists a unique ordered n-tuple $(a_1, \dots, a_n) \in \mathbb{R}^n$ such that*

$$x = a_1 x_1 + \cdots + a_n x_n.$$

Proof.

($\Rightarrow$) We prove the contrapositive; suppose that the second condition is false, and prove that X is dependent. Let $B \subset X$ be a finite subset whose span contains an element $x \in \operatorname{span}(B)$ which may not be expressed in a unique way as a linear combination from B. Since $x \in \operatorname{span}(B)$, there is at least one way to write it as a linear combination from B; thus the only way the condition can be false is if this expression is not unique.

Suppose that there exist distinct n-tuples $(a_i)_i$ and $(b_i)_i$ such that

$$x = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i.$$

Subtracting yields $\sum_{i=1}^{n} (b_i - a_i) x_i = 0$; but for at least one i, we have $a_i \neq b_i$, so this is a nontrivial dependence relation among the x_i's; thus B is not independent, so neither is X.

($\Leftarrow$) Again we prove the contrapositive. Thus we suppose that X is not independent. Then there exists some vectors $x_1, \dots, x_n \in X$ and some real numbers $a_1, \dots, a_n$, not all zero, such that $\sum_{i=1}^{n} a_i x_i = 0$. Since $0 \in \operatorname{span}\{x_1, \dots, x_n\}$, we see that 0 can be written in more than one way as a linear combination of vectors from X. Thus the second condition is also false. □

Example 18. Let $V = \mathbb{R}^n$ and let $X = \{e_1, \dots, e_n\}$.
Then X is independent.

Example 19. Let $V = \mathcal{M}_{m\times n}$ and let $X = \{M_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n\}$.
Then X is independent.

Example 20. Let $V = \mathcal{P}_n$ and let $X = \{1, x, x^2, \dots, x^n\}$.
Then X is independent.

Proposition 12. *Let V be a vector space and let $v, w \in V$ with $v \neq 0$. Then v and w are linearly dependent if and only if $w = cv$ for some $c \in \mathbb{R}$.*

Proof. Suppose that v and w are linearly dependent. Then there exists $a, b \in \mathbb{R}$, not both zero, such that $av + bw = 0$. If $a = 0$, then $bw = 0$, so $w = 0$ (since $b \neq 0$), whence $w = 0v$. Otherwise $a \neq 0$, and $v = -\frac{b}{a}w$.

On the other hand, if $w = cv$, then $cv - w = 0$ is a nontrival dependence relation, so v and w are linearly dependent. ☐

Example 21. Let $V = \mathcal{D}(I)$, where $I \subset \mathbb{R}$ is an open interval. Let $f, g \in V$. If f and g are linearly dependent, then there exists a constant c such that $f(t) = cg(t)$ for all $t \in I$. Assuming neither f nor g is the zero function, we see that $c \neq 0$. Then $f(t) = 0 \Leftrightarrow g(t) = 0$. Since differentiation is linear, we see that $f'(t) = 0 \Leftrightarrow g'(t) = 0$. This continues for all of the derivatives of f and g.

Turning this around, one sees that if there exists $t \in I$ such that f and g, or any of their derivatives, have the property that one is zero at t and the other is not, then f and g are linearly independent.

Example 22. Let $V = \mathbb{R}^3$ and let

$$v_1 = (1, 2, -3), v_2 = (2, 0, 1), v_3 = (4, -4, 9) \in \mathbb{R}^3.$$

Show that the set $\{v_1, v_2, v_3\}$ is dependent.

Solution. If these vectors are indeed linearly dependent, then they all lie on the same plane in $\mathbb{R}^3$. We may choose any one of them, say v_3, are try to write it as a linear combination of the other two. That is, we want to find $x_1, x_2 \in \mathbb{R}$ such that $v_3 = x_1v_1 + x_2v_2$. Thinking of these vectors as column vectors, the equation we want to solve is

$$\begin{bmatrix} 4 \\ -4 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Setting

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -3 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 4 \\ -4 \\ 9 \end{bmatrix},$$

we see that this is equivalent to solving the matrix equation $Ax = b$. We use Gaussian elimination (this is a relatively easy elimination; do it for practice) to obtain an alternate matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}.$$

Thus $x_1 = -2$ and $x_2 = 3$. ☐

The last example does not completely give a test for linear independence.

Example 23. Let $V = \mathbb{R}^3$ and let

$$v_1 = (1, 2, -3), v_2 = (2, 4, -6), v_3 = (4, -4, 9) \in \mathbb{R}^3.$$

Show that the set $\{v_1, v_2, v_3\}$ is dependent.

Attempt. We try to write v_3 as a linear combination of v_1 and v_2. But it isn't! This is because v_1 and v_2 lie on the same line; so actually v_3 in independent from v_1 or from v_2. However, the set is dependent, because $2v_1 - v_2 + 0v_3 = 0$ is a nontrivial dependence relation. □

We now give a test for linear independence in $\mathbb{R}^m$. Let $X = \{v_1, \ldots, v_n\}$ be a subset of $\mathbb{R}^m$. Form the $m \times n$ matrix A by putting the vectors in columns:

$$A = [v_1 \mid \cdots \mid v_n].$$

If $x = [x_1, \ldots, x_n]^t$ is a column vector in $\mathbb{R}^n$, we have seen that

$$\begin{aligned} Ax &= x_1 A^{(1)} + \cdots + x_n A^{(n)} \\ &= x_1 v_1 + \cdots + x_n v_n; \end{aligned}$$

that is, Ax is a linear combination of the columns of A. Now $Ax = 0$ has a solution other than $x = (0, \ldots, 0)$ if and only if there is a nontrivial dependence relation among the v_i's.

Form matrix Q by performing forward elimination on A, so that Q is in row echelon form; there is an invertible $m \times n$ matrix O such that $Q = OA$. Since $O0 = 0$, we have

$$Ax = 0 \Leftrightarrow OAx = O0 \Leftrightarrow Qx = 0.$$

The solution to $Ax = 0$ is unique if and only if Q has no free columns; otherwise, $Ax = 0$ has a nontrivial (i.e., nonzero) solution, which gives a nontrivial dependence relation among the columns of A.

Example 24. Let $V = \mathbb{R}^3$ and let

$$v_1 = (1, 2, -3), v_2 = (2, 0, 1), v_3 = (4, -4, 9) \in \mathbb{R}^3.$$

Show that the set $\{v_1, v_2, v_3\}$ is dependent.

Solution. Put the vectors in columns of a matrix A, so that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -4 \\ -3 & 1 & 9 \end{bmatrix}.$$

Perform forward elimination on A is arrive at

$$Q = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -12 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since Q has a free column, the vectors are not independent. □

8. Bases

Definition 7. Let V be a vector space and let $X \subset V$.

We say that X is a *basis* for V if

(B1) X spans V;
(B2) X is independent.

Definition 8. Let V be a vector space and let $X \subset V$.

We say that X is a *spanning set* for V if X spans V.

We say that X is a *minimal spanning set* for V if

(M1) X spans V;
(M2) $Y \subsetneq X \Rightarrow \operatorname{span}(Y) \subsetneq V$.

Proposition 13. *Let V be a vector space and let $X \subset V$.*
Then X is a basis for V if and only if X is a minimal spanning set for V.

Proof. To prove an if and only if statement, we prove the implication in both directions. Here it is clearly sufficient to show that **(B2)** is equivalent to **(M2)** in the presence of **(B1)**. Thus suppose that X spans V. We prove the contrapositive in both directions.

($\Rightarrow$) Suppose that X is a spanning set which is not minimal. Then there exists a smaller subset $Y \subsetneq X$ which spans. Let $x \in X \smallsetminus Y$; then x is a linear combination of vectors in Y, which demonstrates the presence of a nontrivial dependence relation in X. Thus X is not independent.

($\Leftarrow$) Suppose that X is a spanning set which is dependent. Then there exists a nontrivial dependence relation in X. This allows us to select some vector $x \in X$ and write it as a linear combination of the other vectors in X; let $Y = X \smallsetminus \{x\}$. By Proposition 7 **(b)**, $Y \subset \operatorname{span}(Y)$; also $x \in \operatorname{span}(Y)$, so $X = Y \cup \{x\} \subset \operatorname{span}(Y)$. Thus by Proposition 7 **(e)**,

$$V = \operatorname{span}(X) \subset \operatorname{span}(Y) \subset V,$$

which shows that Y spans V. Thus X is not a minimal spanning set. □

Example 25. Let $V = \mathbb{R}^3$ and let $W = \{(x, y, z) \in V \mid x + y + z = 0\}$. We have seen that $W \leq V$, so W is a vector space. Actually, W is a plane through the origin. Let $v_1 = (1, 0, -1)$ and $v_2 = (0, 1, -1)$ and let $X = \{v_1, v_2\}$. Then X spans W: if $v = (x, y, z) \in W$, then $v = (x, y, -x - y)$, so $v = xv_1 + yv_2$. However, if we remove either vector from the set, the span of what remains is a line. Thus this set is a minimal spanning set, and so it is a basis.

Proposition 14. *Let V be a vector space and let $X \subset V$.*
Then X is a basis for V if and only if every element of V can be written as a linear combination from X in a unique way.

Proof. Exercise. □

Proposition 15. *Let V be a vector space and let $X \subset V$ be independent. If $v \in V \smallsetminus \operatorname{span}(X)$, then $X \cup \{v\}$ is independent.*

Proof. Exercise. □

Proposition 16. *Let V be a vector space and let $X \subset V$ be a spanning set. If $v \in V \smallsetminus X$, then $Y = X \cup \{v\}$ is dependent.*

Proof. If $v = 0$, there result is immediate. Otherwise, we may write v as a linear combination of elements in X, so the expression of v as a linear combination of elements in Y is not unique; thus Y is dependent. □

Proposition 17. *Let V be a vector space and let $X = \{x_1, \ldots, x_n\}$ be a dependent set. Then there exists $k \in \{1, \ldots, n\}$ such that x_k is a linear combination from $\{x_1, \ldots, x_{k-1}\}$.*

Proof. Since X is dependent, there is a nontrivial dependence relation

$$\sum_{i=1}^{n} a_i x_i = 0,$$

where not all a_i's equal zero. Let k be the largest integer between 1 and n such that $a_k \neq 0$. Then

$$x_k = \frac{1}{a_k} \sum_{i=1}^{k-1} a_i x_i$$

is a linear combination of the preceding elements. □

Remark 14. Suppose V and X are as above, and note that x_n is not necessary dependent on the preceding elements. For example, perhaps $V = \mathbb{R}^3$ and $x_1, \ldots, x_{n-1}$ all lie on the same plane, but x_n is perpendicular to it.

Theorem 1. *Let V be a finitely generated vector space and let $X, Y \subset V$. If X is independent and Y spans, then $|X| \leq |Y|$.*

Proof. By Proposition 9, we may assume that Y is finite, say $Y = \{y_1, \ldots, y_n\}$.

By way of contradiction (BWOC), suppose that $|X| > n$ and let

$$Z = \{z_1, \ldots, z_{n+1}\} \subset X;$$

then Z is independent by Proposition 10. Label the elements of Y and Z so that all of those contained in $Y \cap Z$ are in the front:

$$Y = \{z_1, \ldots, z_i, y_{i+1}, \ldots, y_n\}.$$

By Proposition 16, the set

$$\{z_1, \ldots, z_{i+1}, y_{i+1}, y_{i+2}, \ldots, y_n\}$$

is dependent. By Proposition 17, one of these vectors is dependent on the preceding ones, and since the $z_i's$ are linearly independent, there exists $k \in \{i+1, \ldots, n\}$ such that y_k is a linear combination of $\{z_1, \ldots, z_{i+1}, y_{i+1}, \ldots, y_{k-1}\}$. Thus if we remove y_k from the set, it will still span:

$$\text{span}\{z_1, \ldots, z_{i+1}, y_{i+1}, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n\} = V.$$

Continuing in this way, adding the next z and removing a y, we see that after $n - i$ replacements we have

$$\text{span}\{z_1, \ldots, z_n\} = V.$$

Thus the set $Z = \{z_1, \ldots, z_n\} \cup \{z_{n+1}\}$ is dependent by Proposition 16, producing a contradiction. □

Remark 15. There is an alternate proof of this proposition. Let Z and Y be as in the above proof. Since Y spans, we have

$$z_j = \sum_{i=1}^{n} a_{ij} y_i \quad \text{for some } a_{ij} \in \mathbb{R}.$$

One may use Gaussian elimination to solve this system of linear equations to obtain a dependence relation among the z's. However, for this to be used in a rigorous proof, one must first give a formal demonstration that Gaussian elimination works in general.

Corollary 1. (Finite Dimension Theorem)
Let V be a finitely generated vector space. Let X and Y be bases for V. Then $|X| = |Y|$.

Proof. Exercise. □

Example 26. Let $V = \mathbb{R}^n$ and let $X = \{e_1, \dots, e_n\}$.
Then X is a basis for V.

Example 27. Let $V = \mathcal{M}_{m\times n}$ and let $X = \{M_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n\}$.
Then X is a basis V.

Example 28. Let $V = \mathcal{P}_n$ and let $X = \{1, x, x^2, \dots, x^n\}$.
Then X is a basis $\mathcal{P}_n$.

Proposition 18. *Let V be a finitely generated vector space and let $Y \subset V$ be a spanning set. Then there exists a subset $X \subset V$ with $X \subset Y$ such that X is a basis for V.*

Proof. Since Y is a spanning set, Y contains a minimal spanning set, say X, which can be obtained simply by throwing out dependent vectors until none are left. Then X is a basis by Proposition 13. $\square$

Remark 16. In particular, every finitely generated vector space has a basis.

Proposition 19. *Let V be a finitely generated vector space and let $X \subset V$ be independent. Then there exists a subset $Y \subset V$ with $X \subset Y$ such that Y is a basis for V.*

Proof. If X spans V, we are done.

Otherwise, there exists a vector v which is in V but not in $\mathrm{span}(X)$. Form the set $X \cup \{v\}$; this set is still independent by Proposition 15. Continue this process until the resulting set spans; this will happen in a finite number of steps since V is finitely generated. $\square$

Definition 9. Let V be a finitely generated vector space and let $X \subset V$ be independent.

A *completion* of X is a basis Y for V such that $X \subset Y$.

Example 29. Select any two vectors $v_1, v_2 \in \mathbb{R}^3$ that do not lie on the same line. Then the set $X = \{v_1, v_2\}$ is independent. Let v_3 be any vector in $\mathbb{R}^3$ which does not lie on the plane spanned by X. Then $X \cup \{v_3\}$ is a basis for $\mathbb{R}^3$.

9. Dimension

Definition 10. Let V be a vector space.

The *dimension* of V is the smallest cardinality of a spanning set for V, and is denoted by $\dim(V)$. The dimension of the trivial vector space is defined to be zero: $\dim\{0\} = 0$. If $\dim(V) \in \mathbb{N}$, we say that V is *finite dimensional.*

Remark 17. If V is finitely generated, then Corollary 1 tells us that the dimension of V is the number of elements in *any* basis for V. We see that V is finite dimensional if and only if V is finitely generated.

Example 30. The dimension of $\mathbb{R}^n$ is n.

Example 31. The dimension of $\mathcal{M}_{m\times n}$ is mn.

Example 32. The dimension of $\mathcal{P}_n$ is $n+1$.

Example 33. The vector space $\mathcal{F}(I)$ is NOT finite dimensional.

Proposition 20. *Let V be a vector space. Then V is finite dimensional if and only if every independent subset is finite.*

Proof. If V is finite dimensional, we already know that the cardinality of any independent set is less than or equal to the dimension of V.

Suppose V is not finite dimensional; then V is not finitely generated. Let $X \subset V$ be finite and independent. Then X does not span V, so there exists a vector $v \in V \smallsetminus \operatorname{span}(X)$. The set $X \cup \{v\}$ is still independent by Proposition 17. We have taken an arbitrary independent set and produced a bigger one; continuing in this way we obtain an infinite independent set. $\square$

Proposition 21. *Let V be a finite dimensional vector space and let $W \leq V$. Then W is finite dimensional, and* $\dim(W) \leq \dim(V)$.

Proof. Suppose that W is not finite dimensional. Then W has an independent subset of every cardinality. In particular, it has one whose cardinality is larger than the dimension of V, which contradicts Theorem 1. Thus W is finite dimensional, so W has a basis; this basis is a linearly independent subset of V, so its cardinality, which is the dimension of W, must be less than or equal to the dimension of V, again by Theorem 1. $\square$

Proposition 22. *Let V be a finite dimensional vector space and let $U, W \leq V$. Then*

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Proof. Let $\dim(U) = p$, $\dim(W) = q$, and $\dim(p+q) = n$.

Let $X = \{x_1, \ldots, x_n\}$ be a basis for $U \cap W$. We complete this to a basis $Y = \{x_1, \ldots, x_n, u_1, \ldots, u_{p-n}\}$ for U and $Z = \{x_1, \ldots, x_n, w_1, \ldots, w_{q-n}\}$ for W. We see that $B = \{x_1, \ldots, x_n, u_1, \ldots, u_p, w_1, \ldots, w_q\}$ spans $U+V$. But this is an independent set. To see this, let $a_1, \ldots, a_n, b_1, \ldots, b_p, c_1, \ldots, c_q \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{p} b_j u_j + \sum_{k=1}^{q} c_k w_k = 0.$$

Then

$$\sum_{j=1}^{p} b_j u_j = -\sum_{i=1}^{n} a_i x_j - \sum_{k=1}^{q} c_k w_k.$$

The sum on the left is in U and the sum on the right is in W, so the sum on the left is actually in $U \cap W$. Thus we have $d_1, \ldots, d_n$ such that

$$\sum_{j=1}^{p} b_j u_j = \sum_{i=1}^{n} d_i x_i.$$

Since Y is a basis for U, we see that

$$b_1, \ldots, b_p = 0.$$

Similarly the c_k's are all zero, whence the a_i's are all zero. □

Corollary 2. *Let V be a vector space and let $U \leq V$. Then $U = V$ if and only if $\dim(U) = \dim(V)$.*

Proof. Exercise. □

Example 34. Let $V = \mathbb{R}^3$.
Let $U = \operatorname{span}\{(1,2,0),(2,1,0)\}$, and $W = \operatorname{span}\{(1,0,2),(2,0,1)\}$. We see that U is the xy-plane and W is the xz-plane. The sum of U and W is all of $\mathbb{R}^3$. Their intersection is the x-axis. We see that

$$\dim(U+W) = 3 = 2+2-1 = \dim(U) + \dim(W) - \dim(U \cap W).$$

The proof above indicates that we can change our bases for U and W:
$U = \operatorname{span}\{(1,0,0),(2,1,0)\}$ and $W = \operatorname{span}\{(0,0,1),(2,0,1)\}$, so that the union of these bases is a basis for $U+W$.

10. Exercises

Exercise 1.
FB §3.1 # 1, 2, 3, 7, 8, 9, 12, 14, 15, 17
FB §3.2 # 3, 4, 7, 8, 11, 13, 16, 24
FB §3.2 # 27 through 39 (all are pretty good)

Exercise 2. Let $V = \mathbb{R}^3$.
Set $v_1 = (1, -3, 2)$, $v_2 = (3, -7, -1)$, and $v_3 = (1, 1, 12)$.
Show that the set $\{v_1, v_2, v_3\}$ is linearly dependent.

Exercise 3. Let V be a vector space. Let $v \in V$ and $a \in \mathbb{R}$.
Show that $av = 0_V \Rightarrow a = 0$ or $v = 0_V$.

Exercise 4. Let V be a vector space and let $X \subset V$ be independent.
Show that if $v \in V \setminus \operatorname{span}(X)$, then $X \cup \{v\}$ is independent.

Exercise 5. Let V be a finitely generated vector space.
Let X and Y be bases for V.
Show that $|X| = |Y|$.
(Hint: use Theorem 1.)

Exercise 6. Let V be a vector space and let $U \leq V$. Show that $U = V$ if and only if $\dim(U) = \dim(V)$.

Exercise 7. Let $V = \mathbb{R}^3$ and let $a_1, a_2, a_3 \in \mathbb{R}$. Set

$$W(a_1, a_2, a_3) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a_1x_1 + a_2x_2 + a_3x_3 = 0\}.$$

(a) Show that $W(a_1, a_2, a_3) \leq V$.
(b) Show that the general solution to the matrix equation

$$\begin{bmatrix} 1 & 5 & 4 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a subspace of V.

Department of Mathematics, University of California, Irvine
E-mail address: pbailey@math.uci.edu